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On Preference, Freedom and Diversity

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Abstract

We study the problem of ranking sets of options in terms of freedom of choice. We propose a framework in which both the diversity of the options and the preferences of the agent over the options do play a role. We formulate some axioms that re°ect these two aspects of freedom and we study their logical implications. Two di®erent criteria for ranking sets are characterized, which generalize some of the rankings proposed so far in the literature.

Keywords: Ranking Sets; Freedom of Choice; Diversity relations; Potential preferences.

JEL classi¯cation: D71.

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1 Motivation

Recent research in the field of normative economics has been characterised by various attempts to depart from the standard welfaristic framework and move towards wider foundations based on non-utilitarian values. One of the most active research programmes in this area aims at introducing liberty as an intrinsically relevant principle in the evaluation of alternative states of affairs. The interpretations of freedom advanced in this literature revolve around the idea that to be free an agent has to enjoy access to options to choose from. Hence the problem of measuring a person's freedom is handled by finding a suitable measure of the set of options she faces. There is now an extensive literature concerned with the measurement of freedom. A non exhaustive list would include Arrow (1995), Bossert et al. (1994), Jones and Sugden (1982), Klemisch-Ahlert (1993), Pattanaik and Xu (1990, 1998, 1998a, 2000), Puppe (1996), Sen (1991, 1993), Suppes (1987). Surveys of this literature may be found in Barbera', Bossert and Pattanaik (2001) and Peragine (1999).

The analytical framework generally used in this literature is the following: X (with generical element $x; y; z$; etc.) is a finite set of all conceivable objects of concern for any individual, and $P(X)$ (with generic elements $A; B; C$; etc.) denotes the set of all non-empty subsets of X. An element of X is an opportunity or an option, and an element of $P(X)$ is an opportunity set. The problem is that of ranking opportunity sets on the basis of freedom of choice. Hence, let $\succsim$ be a binary relation over $P(X)$, interpreted as "offers at least as much freedom as", with $\sim$ and $\succ$ being respectively the symmetric and asymmetric components of $\succsim$. For any set A, its cardinality is denoted by $\#A$.

A starting point is the article by Pattanaik and Xu (1990): they characterize axiomatically a criterion for ranking opportunity sets in terms of freedom of choice, introducing three axioms as follows. First (Indifference), every unit set should be freedom-wise indifferent to every other: for all options $x; y \in X; \{x\} \sim \{y\}$: Second (Monotonicity), more opportunity should mean more freedom: $\forall x \neq y \in Z; \{x; y\} \succ \{x\}$; third (Independence), if a set A is judged to give at least as much freedom as another set B, then that ranking will be unaffected by the addition to or subtraction from each of an alternative x not contained in either: for all $A; B \in P(X)$ and for all $x \in X - \{A \cup B\}; A \succsim B$ if and only if $A \cup \{x\} \succsim B \cup \{x\}$: These axioms yield the simple cardinality result (Pattanaik and Xu 1990): for all $A; B \in P(X)$; $\succsim$ satisfies the axioms Indifference, Monotonicity and Independence if and only if

$$A \succeq B \Leftrightarrow \#A \geq \#B:$$

This criterion, which applies only to ¯nite opportunity sets, ranks them according to the number of options they contain. It is a quantity-based ranking rule, in which there is no role for information about the nature or the value of di®erent alternatives. This cardinality-based rule is not supported by the authors: rather they judged it as a trivial rule.

A ¯rst explanation of the triviality result is given by Sen (1990, 1991, 1993): the root of the problem is that we ¯nd it absurd to dissociate the extent of our freedom from our preferences over the alternatives. According to Sen's view, the axiomatic structure of Pattanaik and Xu (1990) fails to capture that aspect of freedom linked to the possibility of choosing what is valuable to us. This is what Sen calls the `opportunity aspect' of freedom. >From this point of view, an individual is free if she has access to alternatives that she regards as valuable in terms of some criteria. These criteria may be her own actual preferences - as Sen seems to suggest - or, alternatively, a given set of potential preferences.

Jones and Sugden (1982) ¯rst suggested the use of potential preferences in assessing a person's freedom, and interpreted the potential preferences as the preferences of a \reasonable" person. For them, the intrinsic value of freedom of choice should be judged not in terms of the preferences that the agent actually has, nor in terms of his future preference ordering, but in terms of the preference orderings that a reasonable person can possibly have. According to them, for instance, if any reasonable person would be indifferent between two particular alternatives, then o®ering the choice between these two alternatives to any person would contributes little to her freedom of choice. Pattanaik and Xu (1998) build on Jones and Sugden's idea of reasonable preferences and construct a ranking consistent with it. In comparing two opportunity sets, A and B, Pattanaik and Xu (1998) concentrate on $\max(A)$ and $\max(B)$, where $\max(A)$ is the set of all alternatives in A which reasonable persons may choose from the feasible set A; and similarly for B. The model they propose has the virtue of capturing the opportunity aspect of freedom - i.e., the value of the di®erent alternantives - without collapsing into an indirect utility ranking[1].

[1] A di®erent approach is proposed by Romero-Medina (2000): he ¯rst considers the set $\max(X)$ of all alternatives in the universal set X that reasonable people will choose if the universal set was feasible. Then he concentrates on $\max(X) \cap A$ and $\max(X) \cap B$ when comparing A and B.

The reasonabless criterion to de¯ning the set of potential preferences has been criticised, among others, by Bavetta and Peragine (2000) and by Bavetta and Guala (2001). Bavetta and Peragine (2000), speci¯cally, while agreeing with the idea that freedom should be assessed by looking at options and potential preferences, nonetheless they settle for an alternative point of view on potential preferences, according to which all and only the preference relations available to an agent in a given choice situation should be taken into account in the assessment of her own freedom. Hence they admit cases in which not only the set of options di®er from one individual (one situation) to the other, but also the set of available preference relations. They de¯ne an opportunity situation as a pair composed by the set of options available to a decision maker and the set of potential preference orderings that she is confronting in a speci¯c choice situation. Then, in comparing two opportunity situations, they say that (Bavetta and Peragine 2000, Theorem 2) an individual i enjoys more freedom than another individual j if and only if the choice set that his preference pro¯les elicit from his own opportunity set A has at least as many elements as the choice set that j can elicit by means of his own preference pro¯les from B. This approach, apart from the speci¯c interpretation advanced by Bavetta and Peragine, is very general from an analytical viewpoint, in that it contains, as special case, the situation where the sets of potential preferences available to agents are all the same - i.e., the case considered by Pattanaik and Xu (1998). Therefore, in dealing with the role of preferences in assessing a person's freedom, this last approach will be followed in the present paper.

A di®erent explanation of the triviality of the pure cardinality rule is suggested by the authors themselves (Pattanaik and Xu, 1990): it is not necessarily dependent on the dissociation of the extent of freedom from the preferences over the alternatives. Instead, they put the blame on the last, independence axiom: it does not take into account the extents to which the di®erent alternatives are \close" or similar to each other. Thus, a development along this line of criticism amounts to introduce formally a notion of closeness or similarity of options, and to reformulate the independence axiom. This is the direction undertaken by the same authors in a subsequent paper (Pattanaik and Xu, 2000) and, more recently, by Bossert et al. (2001).

To sum up, the axiomatic structure of Pattanaik and Xu can be questioned on the basis of two considerations. First, it does not consider the fundamental link between freedom and preferences, therefore failing to capture the \opportunity" aspect of freedom. Second, as the authors point out, it does not take into account the \similarity" or \closeness" aspect of the available alternatives; this is the "variety" aspect of freedom. These two considerations motivate the basic directions of research undertaken in the literature on the measurement of freedom in the last years.

However, each of the two questions has been addressed very much in isolation from the other. The main purpose of this paper is to provide a unified framework for the analysis of the two themes.

The basic ingredients of our framework are the following: (i) a notion of similarity which can be represented by means of a binary similarity relation on the universal set of options (see Pattanaik and Xu, 2000), and (ii) a set of preference relations over the options, interpreted along the lines of Bavetta and Peragine (2000).

We propose some axioms that capture both the variety and the opportunity aspect of freedom, and study their logical implications. Various criteria for ranking opportunity sets are characterized, which generalize some of the ranking proposed so far in the literature.

2 The analytical framework

2.1 Notation

We are denoting by X the universal finite set of opportunities, by $\mathcal{P}(X)$ the set of opportunity sets, and by $N = \{1, \ldots, n\}$ the set of agents.

We first introduce information about the individual preferences over the opportunities.

A decision maker's preference relation is denoted by R. R is a complete, reflexive and transitive binary relation defined over X. The set of all possible preference profiles is denoted by $\Pi = \{R_1, \ldots, R_m\}$. Thus, $\forall x, y \in X$; $\forall h \in \{1, \ldots, m\}$; xR_hy means that "x is preferred to y according to the preference ordering R_h": In our framework each of the n individuals in the society holds a set Π_i; $i \in N$; of preference orderings, where $\Pi_i \subseteq \Pi$. $\mathcal{P}(\Pi) = 2^{\Pi} - \{\emptyset\}$ represents the set of all subsets of preference profiles.

As explained in the previous sections, we are interested in ranking opportunity situations. An opportunity situation is a pair $(A; \Pi_i) \in \mathcal{P}(X) \times \mathcal{P}(\Pi)$. Hence our ranking is represented by a binary relation $\succeq$ over $\mathcal{P}(X) \times \mathcal{P}(\Pi)$. The expression $(A; \Pi_i) \succeq (B; \Pi_j)$ should be read as "the opportunity situation $(A; \Pi_i)$ offers at least as much freedom as the opportunity situation $(B; \Pi_j)$".

We then introduce the following definition: $\forall (A; \Pi_i) \in \mathcal{P}(X) \times \mathcal{P}(\Pi)$;

we define the choice set as:

$$\max_i(A) = \{x \in X : \nexists y \in A \text{ such that } yRx \text{ for some } R \in \Pi_i\}.$$

We now introduce the information about similarity of alternatives.

Let S be a reflexive and symmetric binary relation over X : xSy means that x is similar to y; $x\not{S}y$ means that x is not similar to y. For all $A \in P(X)$, A is homogeneous if and only if, $\forall a, a' \in A, aSa'$. For all $A \in P(X)$, a similarity based partition of A is defined as a class $\{A_1, \ldots, A_m\}$ such that: (1) $A_k \subseteq A, \forall k = 1, \ldots, m$; (2) $\cup_{k=1}^{m} A_k = A$; (3) $A_1, \ldots, A_m$ are pairwise disjoint; (4) A_k is homogeneous $\forall k = 1, \ldots, m$. The similarity based partition is denoted by $\phi(A), \phi'(A), \phi''(A)$, etc. Let $\Phi(A)$ be the set of all similarity partitions $\phi(A)$ of A such that, for every similarity partition $\phi'(A)$, $\#\phi'(A) \geq \#\phi(A)$. So, $\Phi(A)$ is the set of all smallest similarity based partitions of A. For all $x \in X$ and all $A \in P(X)$ we say that xSA if and only if xSa $\forall a \in A$. For all $A, B \in P(X)$, with A homogeneous, we say that A does not mimic B if and only if, for all $\phi(B) \in \Phi(B)$ there exists $a \in A$ such that, for all $B_i \in \phi(B)$, it is not true that aSB_i.

Next we impose some axioms on $\succeq$, which capture our intuition on the extent of freedom enjoyed by the decision maker under alternative situations. We assume first that the relation $\succeq$ is transitive.

3 The axioms

In this section we consider several axioms that defines properties of the binary relation $\succeq$ over X. We divide these axioms in three different groups: Preference based, Similarity based and constructive axioms. This classification has to do with the objective of our paper of characterizing binary relations that embodies both the opportunity aspect of freedom and the variety aspect.

3.1 Preference based axioms

Axiom 1 Indifference between no-freedom situations (INF).

$\forall A, B \in P(X); \forall \Pi_i, \Pi_j \in P(\Pi)$; $[\max_i(A) = \{x\}$ and $\max_j(B) = \{y\}] \Rightarrow (A; \Pi_i) \sim (B; \Pi_j)$.

According to this version of indifference between no-freedom situation, if two opportunity situations defined over meaningful choices lead to as many choice sets that are singletons, then the degree of freedom that they offer is the same.

Axiom 2 Inclusion monotonicity (IMON) $\forall A, B \in P(X); \forall \sigma_i, \sigma_j \in P(\sigma); \phi(A) = A_k$ and $\phi(B) = B_s$ if $A_k \subseteq B_s$ and $\max_i(A_k) \setminus B_s \neq \emptyset$, then $(A; \sigma_i) \succ (B; \sigma_j)$. If $A_k \subseteq B_s$ and $\max_i(A_k) \setminus B_s = \emptyset$, then $(A; \sigma_i) \sim (B; \sigma_j)$:

IMON adapts a preference independence axiom, one that Sen (1991) called weak dominance, to the present context. IMON requires that set inclusion of relevant alternatives implies preference. It restricts the role of non-relevant alternatives and excludes them from consideration when a set is compared with one of its subsets.

3.2 Similarity based axioms

Axiom 3 Indi®erence between Similar situations (ISS).
$\forall A, B \in P(X); \forall \sigma_i, \sigma_j \in P(\sigma); \phi(A) = A_k$ and $\phi(B) = B_s] \Rightarrow (A; \sigma_i) \sim (B; \sigma_j)$.

Axiom ISS is in spirit similar to the principle of no choice situation introduced by Jones and Sugden (1982) and used by Pattanaik and Xu (1990). This principle considers that if two opportunity set are singleton, then de degree of freedom o®ered by them is identical. In ISS this indi®erence situation is attributed to sets where all the alternatives are similar according to the similarity relation de¯ned. It is important that preferences are playing no role on the de¯nition of this axiom.

Axiom 4 Similarity Monotonicity (SM). $\forall A \in P(X)$; A homogeneous, $\forall \sigma_i \in P(\sigma); \forall x \in X - A; [xSa; \forall a \in A] \Rightarrow (A \cup \{x\}; \sigma_i) \sim (A; \sigma_i)$ and $\neg[xSa; \forall a \in A] \Rightarrow (A \cup \{x\}; \sigma_i) \succ (A; \sigma_i)$:

The SM axiom was introduced in Pattanaik and Xu (2000). This axiom explicitly takes into account information about similarity of alternatives in the simple cases involving freedom comparisons of an existing set A in which all the elements in A are similar to each other and an enlarged set $A \cup \{x\}$ where x is outside of A.

> In choosing di®erent modes of transport , if the option of traveling by a red bus is similar to the option of traveling by a blue bus, then it is plausible to argue that the set {blue bus , red bus} o®ers the same amount freedom as the set {red bus}; and if traveling by a red bus is not similar to traveling by a red train, then it is reasonable to argue that the set {red bus , red train} o®ers more freedom than the set {red bus}.

SM thus formally requires that, given a homogeneous set A and $x \in X \setminus A$, if x is similar to all the elements in A, then the addition of x to A does not change the degree of freedom already offered by the opportunity set A, and if x is dissimilar to at least one element in A, then, adding x to A will actually increase the degree of freedom.

3.3 Constructive axioms

Axiom 5 Similarity Composition (SC). $\forall A, B, C, D \in P(X); \forall \Pi_i, \Pi_j \in P(\Pi)$, such that $B \cap D = A \cap C = \emptyset$; C and D are homogeneous and C does not mimic A;

$$[(A; \Pi_i) \sim (B; \Pi_j) \text{ and } (C; \Pi_i) \sim (D; \Pi_j)] \Rightarrow (A \cup C; \Pi_i) \sim (B \cup D; \Pi_j).$$

SC is a weaker version, proposed by Pattanaik and Xu (2000), of an axiom proposed originally by Sen (1991). Sen's axiom requires that, given $A \cap C = B \cap D = \emptyset$, if $[A \sim B$ and $C \sim D]$, then $[A \cup C \sim B \cup D]$, and, if $[A \succ B$ and $C \sim D]$, then $[A \cup C \succ B \cup D]$. However, in our context, a modification of Sen's axiom seems to be warranted. Suppose $A = \{a\}$, $B = \{b\}$, $C = \{c\}$ and $D = \{d\}$ and suppose the elements of these four sets are all distinct. Let aSc and $b\not{S}d$, i.e. a and c are similar but b and d are dissimilar. In view of this, we can justifiably feel that adding c to the set $\{a\}$ does not significantly increase the degree of freedom while adding d to the set $\{b\}$ does. In that case, we may be unwilling to accept that $[A \cup C \sim B \cup D]$ even if $[A \sim B$ and $C \sim D]$. SC deals with this problem of Sen's axiom by restricting the applicability of the axiom to the case where both C and D are homogeneous, and C does not mimic A.

Axiom 6 Composition (COM) iff, for all $A, B, C, D \in P(X)$; such that $\max_i(A) \cap \max_i(C) = \max_j(B) \cap \max_j(D) = \emptyset$;

$$[(A; \Pi_i) \succsim (B; \Pi_j) \text{ and } (C; \Pi_i) \succsim (D; \Pi_j)] \rightarrow [(A \cup C; \Pi_i) \succsim (B \cup D; \Pi_j)]; \text{ and;}$$
$$[(A; \Pi_i) \succsim (B; \Pi_j) \text{ and } (C; \Pi_i) \succ (D; \Pi_j)] \rightarrow [(A \cup C; \Pi_i) \succ (B \cup D; \Pi_j)].$$

The COM axiom was originally defined by Sen (1991) and it required $A \cap C = B \cap D = \emptyset$. We adapt the axiom to the context of this paper, by requiring the no intersection condition to hold only for the maximal elements in the relevant sets.

4 The first ranking

We start our exercise by characterizing an ordering which incorporates both information about preferences and the similarity of alternatives using a cardinal approach. First we select the alternatives that, in a given set A, are relevant according to the available preferences $\succsim_i$, i.e., $\max_i(A)$; then we count the number of elements in the similarity based partition of $\max_i(A)$, that is the number of elements contained in $\phi(\max_i(A))$.

Definition 1 $\succeq = \succeq_1$ iff and only if for all $(A; \succsim_i); (B; \succsim_j) \in \mathcal{P}(X) \times \mathcal{P}(\succsim)$,

$$(A; \succsim_i) \succeq (B; \succsim_j) \Leftrightarrow \#\phi(\max_i(A)) \geq \#\phi(\max_j(B))$$

$\forall \phi(A) \in \Phi(A), \forall \phi(B) \in \Phi(B).$

That is, an opportunity situation $(A; \succsim_i)$ offers more freedom of choice than another opportunity situations $(B; \succsim_j)$ if and only if the number of alternatives contained in $\phi(\max_i(A))$ is bigger than the number of alternatives contained in $\phi(\max_j(B))$.

We now characterize the ordering just introduced.

Theorem 1 $\succeq = \succeq_1$ if and only if $\succeq$ satisfies the axioms INF, SM, SC and COM.

Proof. The necessity part of the proposition is straightforward; we prove only the sufficiency part[2]. Hence, let $\succeq$ satisfy INF, SM, SC and COM. We first prove that:

(1) $\forall A \in \mathcal{P}(X), \forall \succsim_i \in \mathcal{P}(\succsim)$, $(A; \succsim_i) \sim (\max_i(A); \succsim_i)$.

If $\max_i(A) = A$, then the result clearly follows. If not, suppose $\#\left(\max_i(A)\right) = g$ and let $\max_i(A) = \{a_1, \ldots, a_g\}$ and $A - \max_i(A) = \hat{A}$. Now, $\max_i\left(\{a_1\} \cup \hat{A}\right) = \max_i(\{a_1\}) = \{a_1\}$ and $\max_i\left(\{a_2\} \cup \hat{A}\right) = \max_i(\{a_2\}) = \{a_2\}$. Hence by INF,

$$\left(\{a_1\} \cup \hat{A}; \succsim_i\right) \sim (\{a_1\}; \succsim_i)$$

and

$$\left(\{a_2\} \cup \hat{A}; \succsim_i\right) \sim (\{a_2\}; \succsim_i).$$

[2] This proof makes extensive use of results in Pattanaik and Xu (2000).

Clearly, $\{a_1\} \cap \{a_2\} = \emptyset$, $\max_i(\{a_1\} \cup \{a_2\}) = (\{a_1\} \cup \{a_2\})$, $\left(\{a_1\} \cup \hat{A}\right) \cap \left(\{a_2\} \cup \hat{A}\right) = \hat{A}$ and $\hat{A} \cap \max_i\left(\left(\{a_1\} \cup \hat{A}\right) \cup \left(\{a_2\} \cup \hat{A}\right)\right) = \emptyset$. Hence we can apply axiom COM and obtain,

$$\left(\{a_1\} \cup \{a_2\} \cup \hat{A}; \succsim_i\right) \sim (\{a_1\} \cup \{a_2\}; \succsim_i).$$

By considering successively $a_3; a_4; \ldots; a_g$; and applying INF and COM repeatedly, we finally obtain

$$\left(\max_i(A) \cup \hat{A}; \succsim_i\right) \sim (\max_i(A); \succsim_i)$$

or

$$(A; \succsim_i) \sim (\max_i(A); \succsim_i).$$

By (1) and transitivity, we obtain:

(2) $(A; \succsim_i) \succeq (B; \succsim_j) \Leftrightarrow (\max_i(A); \succsim_i) \succeq (\max_j(B); \succsim_i)$.

Now consider that, according to INF, $(\{x\}; \succsim_i) \sim (\{y\}; \succsim_j)$. This is what required by the axiom INS in Pattanaik and Xu (2000); moreover, axioms SM and SC correspond, respectively, to the S-Monotonicity and the S-composition axioms of Pattanaik and Xu (2000). Therefore, on the basis of Theorem 4.6 of Pattanaik and Xu (2000), we know that, for all $\varphi(A) \in \Pi(A)$ and for all $\varphi(B) \in \Pi(B)$, $(A; \succsim_i) \succeq (B; \succsim_j)$ if and only if $\#\varphi(A) \geq \#\varphi(B)$. Hence we have:

(3) $(\max_i(A); \succsim_i) \succeq (\max_j(B); \succsim_i) \Leftrightarrow \#\varphi(\max_i(A)) \geq \#\varphi(\max_j(B))$.

Finally, (2) and (3) imply:

$$(A; \succsim_i) \succeq (B; \succsim_j) \Leftrightarrow \#\varphi(\max_i(A)) \geq \#\varphi(\max_j(B)).$$

■

5 The second ranking

In this section we characterize a second ordering. This ordering also incorporates both information about preferences and similarity by using a cardinal approach. For a given set A, we first study the similarity based partition $\varphi(A) = \{A_1; \ldots; A_m\}$; then, for each $A_k \in \varphi(A)$, we concentrate on the relevant alternatives, as elicited by the available set of preferences: i.e., on $\max_i(A_k)$; finally, we aggregate the sets $\max_i(A_k)$ for all the $k = 1; \ldots; m$; and we denote the resulting set by $\max_i(\varphi(A))$. Formally,

Definition 2 For all $A \in P(X)$, for all $\Pi_i \in P(\Pi)$, for all $\phi(A) = \{A_1, \ldots, A_m\} \in \Phi(A)$,

$$\max_i(\phi(A)) := \cup_{k=1}^{m} \max_i(A_k)$$

We now define our second ordering:

Definition 3 $\succeq = \succeq_2$ iff and only if for all $(A, \Pi_i), (B, \Pi_j) \in P(X) \times P(\Pi)$,

$$(A, \Pi_i) \succeq (B, \Pi_j) \Leftrightarrow \# \max_i(\phi(A)) \geq \# \max_j(\phi(B))$$

$\forall \phi(A) \in \Phi(A), \forall \phi(B) \in \Phi(B)$.

That is, an opportunity situation (A, Π_i) offers more freedom of choice than another opportunity situations (B, Π_j) if and only if the number of alternatives contained in $\max_i(\phi(A))$ is bigger than the number of alternatives contained in $\max_j(\phi(B))$.

We now characterize the ordering $\succeq_2$.

Theorem 2 $\succeq = \succeq_2$ if and only if $\succeq$ satisfies the axioms ISS, IMON, and COM.

Proof. The necessity part of the proposition is straightforward; we prove only the sufficiency part. This proof has two stages. Let $\succeq$ satisfy ISS, COM and IMON. First, we show that:

$$\text{for all } A, B \in Z, \text{ if } \# \max_i(\phi(A)) = \# \max_j(\phi(B)), \text{ then } (A, \Pi_i) \sim (B, \Pi_j). \tag{1}$$

Suppose $A, B \in P(X)$ and $\# \max_i(\phi(A)) = \# \max_j(\phi(B)) = g$. Let $\max_i(\phi(A)) = \{a_1, \ldots, a_g\}$ and $\max_j(\phi(B)) = \{b_1, \ldots, b_g\}$. By ISS,

$$\{a_1\} \sim \{b_1\} \tag{2}$$

and

$$\{a_2\} \sim \{b_2\}. \tag{3}$$

$\{a_1\} \cap \{b_1\} = \{a_2\} \cap \{b_2\} = \emptyset$ and, further $\max_i(\phi\{a_1\}) = \{a_1\}, \max_i(\phi\{a_2\}) = \{a_2\}$ and $\max_j(\phi\{b_1\}) = \{b_1\}, \max_j(\phi\{b_2\}) = \{b_2\}$, since $a_1, a_2 \in \max_i(\phi(A))$ and $b_1, b_2 \in \max_j(\phi(B))$. Hence by (2), (3) and COM, we have

$$\{a_1, a_2\} \sim \{b_1, b_2\}. \qquad (4)$$

By ISS, again,

$$\{a_3\} \sim \{b_3\}. \qquad (5)$$

By (4), (5) and COM,

$$\{a_1, a_2, a_3\} \sim \{b_1, b_2, b_3\}. \qquad (6)$$

Proceeding in this way, we finally have $\{a_1, \ldots, a_g\} \sim \{b_1, \ldots, b_g\}$; that is $\max_i(\phi(A)) \sim \max_j(\phi(B))$. If $A = \max_i(\phi(A))$, then $A \sim \max_j(\phi(B))$. Suppose $\{A \setminus \max_i(\phi(A))\} \neq \emptyset$. Let $\{A \setminus \max_i(\phi(A)\} = \{a'_1, \ldots, a'_m\} \neq \emptyset$. It is clear that $T_1 = \max_i(\phi(A)) \cup \{a'_1, \ldots, a'_m\}$ is such that $T_1 \subseteq A$ and $\max_i(\phi(A) \setminus T_1 = \emptyset$. Then, by IMON, $T_1 \sim \max_j(\phi(B))$. Hence we have

$$(A, \Pi_i) \sim \max_j(\phi(B)). \qquad (7)$$

Similarly, by IMON, from (7), we have $(A, \Pi_i) \sim B$, which proves (1).

Next, we show:

$$\text{for all } A, B \in Z, \text{ if } \# \max_i(\phi(A)) > \# \max_j(\phi(B)), \text{ then } (A, \Pi_i) \succ (B, \Pi_j). \qquad (8)$$

Suppose $A, B \in \mathcal{P}(X)$ and $\# \max_i(\phi(A)) > \# \max_j(\phi(B))$. Let $\# \max_j(\phi(B)) = g$ and $\# \max_i(\phi(A)) = g + t$ (where $t > 0$). Further, let $\max_i(\phi(B)) = \{b_1, \ldots, b_g\}$ and $\max_i(\phi(A)) = \{a_1, \ldots, a_g, \ldots, a_{g+t}\}$. Note that $\max_i(\phi\{a_1, \ldots, a_g\}) = \{a_1, \ldots, a_g\}$. Hence, by (1),

$$\{a_1, \ldots, a_g\} \sim (B, \Pi_j) \qquad (9)$$

since $\max_i(\phi(A)) = \{a_1, \ldots, a_g, \ldots, a_{g+t}\}$ it is clear that $T_{g+1} = \max_i(\phi\{a_1, \ldots, a_g\}) \cup \{a_{g+1}\}$ is such that $\{a_1, \ldots, a_g\} \subseteq T_{g+1}$ and $\max_i(\phi(T_{g+1}) \setminus \{a_1, \ldots, a_g\} \neq \emptyset$. Then by IMON and (9), it follows that

$$T_{g+1} \succ \{a_1, \ldots, a_g\}$$

and by (9)

$$T_{g+1} \succ \max_j(\phi(B)). \qquad (10)$$

Taking (10), adding $a_{g+2}, \ldots, a_{g+t}$ on the left hand side, and using IMON repeatedly, we have

$$\{a_1, \ldots, a_{g+t}\} \succ (B, \Pi_j). \qquad (11)$$

Taking (11) and using an argument similar to the one used to establish (7), by IMON, we have $(A, \Pi_i) \succ (B, \Pi_j)$, which proves (8). (1) and (8) complete the proof of the sufficiency part of the proposition. ■

According to the rule characterized in theorem 2, given a set A and a similarity partition $\phi(A) = \{A_1, \ldots, A_m\}$, we should concentrate, for each $A_k \in \phi(A)$, on the set of relevant alternatives $\max_i(A_k)$, and we should then aggregate these sets for all $k = 1, \ldots, m$. A different approach, that at first sight could seem appealing, would amount to first concentrating on the relevant alternatives in the overall set, i.e., $\max_i(A)$; then, to focus on the relevant options contained in each similarity class, i.e., $\max_i(A) \cap A_k$; finally, in aggregating the resulting sets over the m class, hence obtaining the set $\cup_{k=1}^{m} (\max_i(A) \cap A_k)$. One moment reflection on this alternative rule, however, reveals that, in so doing, all the information about similarity are lost. In fact, it turns out that $\cup_{k=1}^{m} (\max_i(A) \cap A_k) = \max_i(A)$; hence a ranking based, say, on the cardinality of the set $\cup_{k=1}^{m} (\max_i(A) \cap A_k)$ would be equivalent to a ranking based on the cardinality of $\max_i(A)$. The similarity relation becomes completely irrelevant.

6 Relation with the literature

Before concluding we wish to draw the attention of the reader toward the relationship that exists between our ranking and some of the main results achieved so far in the literature. As the following remarks illustrate, the rules proposed in this paper have the nice property of generalizing some important results so far axiomatized in the literature.

The basic ingredients of our analysis are the similarity relation on the set of alternatives and the set of preferences available to the agents. The following remarks show what happens when these aspects of freedom are deemed to be irrelevant.

We first study the case of irrelevance of the similarity relation:

Remark 1 If $\nexists\ a, b \in X$ such that aSb, then the ranking established in Theorem 1 coincides with the ranking established in Theorem 2 and with the rule characterized by Bavetta and Peragine (2000) (proposition 5.2).

We now turn to the case of irrelevance of the preference relations:

Remark 2 Suppose that the set of preference orderings $\Re$ satisfies a "richness" assumption, such that $\forall A \in \Re(X)$; $\max(A) := \{x \in A : \nexists y \in A$ such that yRx for some $R \in \Re\} = A$. Then, if $\forall i \in N$; $\Re_i = \Re$, all possible preference profiles can be hold by the individuals, and the ranking established in Theorem 1 coincides with the Simple Similarity-based Ordering of Pattanaik and Xu (2000). Moreover, in this case, the ranking established in Theorem 2 coincides with the Simple Cardinality-based Ordering of Pattanaik and Xu (1990).

We now consider the case of irrelevance of the similarity relation and of the preference relations:

Remark 3 Suppose, again, that $\Re$ satisfies the "richness" assumption, such that $\forall A \in \Re(X)$; $\max(A) := \{x \in A : \nexists y \in A$ such that yRx for some $R \in \Re\} = A$. If, in addition to assuming that $\forall i \in N$; $\Re_i = \Re$, we assume that $\nexists\, a, b \in X$ such that aSb; then the ranking established in Theorem 1 coincides with the ranking established in Theorem 2 and with the Simple Cardinality-based Ordering of Pattanaik and Xu (1990).

The final remarks studies the consequences of adopting the "reasonable" preferences view, in the case of irrelevance of the similarity relation.

Remark 4 If $\forall i \in N$; $\Re_i = \Re^*$, where $\Re^*$ stands for the set of reasonable preference profiles à la Jones and Sugden (1982) and Pattanaik and Xu (1998), and, moreover, $\nexists\, a; b \in X$ such that aSb; then the ranking established in Theorem 1 coincides with the rule characterized by Pattanaik and Xu (1998) (proposition 5.1).

7 Conclusion

In this paper we have tried to provide a unified framework for the analysis of the opportunity aspect and of the the variety aspect of freedom. The basic ingredients of our framework are the following: (i) a notion of similarity, represented by means of a binary similarity relation on the universal set of options (see Pattanaik and Xu, 2000), and (ii) a set of preference relations over the options, interpreted along the lines of Bavetta and Peragine (2000).

We have proposed some axioms that capture both the variety and the opportunity aspect of freedom, and studied their logical implications. Two different criteria for ranking opportunity sets have been characterized, where both preferences and diversity of options do play a role. These criteria generalize some of the rankings proposed so far in the literature.

One possible extension of this work would consist in considering a more articulated notion of similarity, along the line of Bossert et al. (2001). This will be the subject of future research.

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